

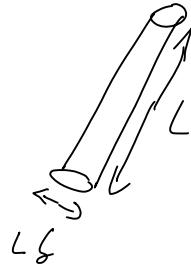
Wolff's Hairbrush - Tim Colliander

$S^{n-1} \subset \mathbb{R}^n$ $\alpha \in S^{n-1}$ "direction"

α -like: any translate of $\{\alpha s \in \mathbb{R}^n : s \in \mathbb{R}\}$

$I_{\alpha, L}$: interval of length L contained in an α -like

The L_δ -bd of $I_{L, \alpha}$ is a δ -eccentric tube of scale L .



A δ -tube T_α could have any scale L . (axis on α -line)

Consider $\mathcal{R} \subset S^{n-1}$

$$\cdot |\mathcal{R}| \sim S^{n-1}$$

$$\cdot \forall \alpha \in S^{n-1} \exists w \in \mathcal{R} \text{ st } |\alpha - w| \leq \delta$$

\mathcal{R} is called δ -separated

E , a Besicovitch set in \mathbb{R}^n

Measuring overlap of δ -tubes

$\sum_{w \in \mathcal{R}} \chi_{T_w}(x)$ measures overlap at x .

Maximum overlap (pointwise)

$$\left\| \sum_{w \in \mathcal{R}} \chi_{T_w}(x) \right\|_{L^\infty} \leq \delta^{1-h}$$

Lossy L^1 estimate

$$\left\| \sum_{w \in \mathcal{R}} \chi_{T_w} \right\|_{L^1} = \sum_{w \in \mathcal{R}} \|\chi_{T_w}\|_{L^1} = \sum_{w \in \mathcal{R}} |T_w|$$

KMOC

Suppose \mathcal{R} is δ -separated, $\{T_w\}_{w \in \mathcal{R}}$ ^{δ -type collection}

$$\begin{aligned} \left\| \sum_{w \in \mathcal{R}} \chi_{T_w} \right\|_{L^{\frac{n}{n-1}}} &\leq \underbrace{\delta^{-\varepsilon}}_{n} \left(\sum_{w \in \mathcal{R}} |T_w| \right)^{\frac{n-1}{n}} \\ &\stackrel{\forall \varepsilon > 0 \exists C_\varepsilon}{\leq} C_\varepsilon \delta^{-\varepsilon} \end{aligned}$$

Interpolation Heuristic

Set $L=1$, we can ignore $\sum_{w \in \mathcal{R}} |T_w|^{\sim 1}$
(interpolation)

$$(\text{target} - L^p) \left\| \sum_{w \in \Omega} x_{Tw} \right\|_p \leq \zeta^{-\varepsilon} \zeta^{\frac{n}{p} - (n-1)} \\ \frac{n}{n-1} \leq p \leq \infty$$

Fact : (Target L^p) $\Rightarrow d(n) \geq \frac{p}{p-1}$ (Mausdorff, Minkowski)

$$\text{So } p = \frac{n}{n-1} \Rightarrow d(n) \geq n$$

History

Target L^p

$d(n) \geq$

Condobu '77

$$p=2 (n=2)$$

2

Duniry '83

$$\frac{n+1}{2}$$

Boungah '91

$$p = \frac{n+1}{n-1}$$

"

Wolf '95

$$p = \frac{n+2}{2}$$

$$\frac{n+2}{2}$$

Corboda: bilinear + angle scales $\Rightarrow p=2$ ($n=2$)

$\Omega, \{T_w\}, (n=2)$

$$\left\| \sum_{w \in \mathcal{R}} \chi_{T_w} \right\|_{L^2} \leq \left(\log \frac{1}{\delta} \right)^{\gamma_2} \left(\sum_{w \in \mathcal{R}} |T_w| \right)^{\gamma_2}$$

Proof: ① Bilinearize: $\left\| \chi_{T_w} \right\|_{L^2} = \left| \sum_w \chi_{T_w} \sum_{w'} \chi_{T_{w'}} dx \right|$

$$= \sum_w \sum_{w'} |T_w \cap T_{w'}|$$

It suffices to show

$$\sum_{\substack{w' \in \mathcal{R} \\ (w' \neq w)}} |T_w \cap T_{w'}| \leq \log \frac{1}{\delta} |T_w|$$

② Geometry of angle scales.

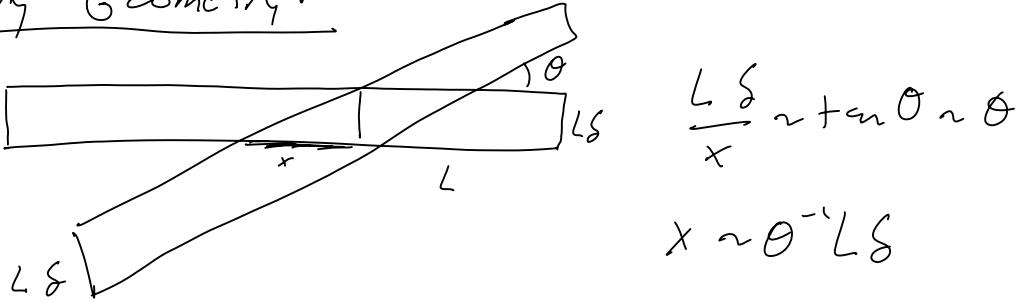
Suppose w, w' differ by angle $\sim 2^{-k}$ some k satisfying $\delta \leq 2^{-k} \leq 1$

Elementary Geometry $\Rightarrow |T_w \cap T_{w'}| \leq 2^k \delta |T_w|$

$$\sum_{k=0}^{\log \frac{1}{\delta}} \sum_{\substack{w \in \mathbb{R} \\ L(w, w') \sim 2^{-k}}} 2^k \delta |T_w| \leq \log \frac{1}{\delta}$$

\cap (essentially arctic circle)

Elementary Geometry 1



Dvury showed $d(\eta) \geq \frac{n+1}{2^n}$
 E Besicovitch set in \mathbb{R}^n

$\forall w \in S^{n-1}/(\text{antipodal}) \exists$ line segment

$$l_w: [0, 1] \rightarrow E$$

Consider $\phi: (S^{n-1}/\pm)_w \times [0, 1]_s \times [0, 1]_t \rightarrow E \times E \setminus P$

$$\phi(w, s, t) = (l_w(s), l_w(t))$$

1-1: Any two nonparallel lines intersect at most one pt.

$$n-1+1+1 = \dim(\text{domain}[\psi]) = \dim(\text{Range}[\psi]) = 2\dim E \\ = n+1$$

$$\text{Bourgain's Bush} \Rightarrow p = \frac{n+1}{n-1}$$

Bourgain's idea: maximum overlap requires an x within intersection of all the tubes.

- Away from this multiplicity point, tubes are essentially disjoint



- A key idea in the implementation is to look at

$$E_K := \left\{ x : \sum_{w \in \Omega} \chi_{T_w} \sim 2^K \right\}; \quad 1 \leq 2^K \leq \delta^{1-n}$$

$$\#K \sim \log \frac{1}{\delta}$$

$$\underline{\text{Wolff's Hairbrush}} \Rightarrow p = \frac{n+2}{2} \xrightarrow{\text{fact}} d(n) \geq \frac{n+2}{2}$$

(5) Bilinearize (but $\frac{p}{2} < 1$ so triangle inequality?)

pseudo-triangle inequality:

$$a+b \leq (a^p + b^p)^{\frac{1}{p}} \quad p \leq 1$$

$$(\sum a_i)^p \leq \sum a_i^p$$

$$\left\| \sum_{i=1}^n f_i \right\|_p \leq \left(\sum_{i=1}^n \|f_i\|_p^p \right)^{\frac{1}{p}}$$

$$\sum_{K=0}^{\log \frac{1}{\delta}} \left\| \sum_{\substack{w, w' \\ |w-w'| \sim 2^{-K}}} \chi_{T_w} \chi_{T_{w'}} \right\|_{p/2}^{p/2} \lesssim \delta^{n-p(n-1)} \quad (*)$$

② clever rescaling shows $K=0 \Rightarrow K>1$ OK

③ imitate Bourgault but in a bilinear way

$$E_{\mu, \mu'} = \left\{ x : \sum_{w \in \mathcal{R}_1} \chi_{T_w}(x) \sim \mu ; \sum_{w \in \mathcal{R}_2} \chi_{T_{w'}}(x) \sim \mu' \right\}$$

On w -tse, get high multiplicities on tube from $T_{w'}$ tubes from (*)